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Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition ☆

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Abstract

In this paper we investigates the blow-up properties of the positive solutions to a porous medium equation with nonlocal reaction source and with nonlocal boundary condition, we obtain the blow-up condition and its blow-up rate estimate.

Keywords: Porous medium equation; Nonlocal reaction sources; Nonlocal boundary condition; Blow-up; Blow-up rate

1. Introduction

In this paper we study the following porous medium equation with a nonlocal source subject to a weighted nonlocal boundary condition:

$$\begin{cases} u_t = \Delta u^m + au^q \displaystyle\int_\Omega u^p(y,t)\,dy, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\,dy, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$, $m > 1$, $p > 0$ and $q \geqslant 0$. $\varphi(x,y)$ in the boundary condition is continuous, nonnegative on $\partial\Omega \times \overline{\Omega}$ and not identically zero, $\int_\Omega \varphi(x,y)\,dy > 0$ on $\partial\Omega$. The initial data

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$u_0 \in C^{2+\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$, $u_0 \geqslant 0$, and satisfies the compatibility condition $u_0(x) = \int_\Omega \varphi(x, y)u_0(y)\,dy > 0$ for $x \in \partial\Omega$.

A function $u(x,t)$ is called a classical solution of problem (1.1) if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T))$ for some T, $0 < T \leqslant +\infty$, and satisfies (1.1). If $T = +\infty$, $u(x,t)$ is called a global solution of (1.1).

In the past several decades, many physical phenomena were formulated as nonlocal mathematical models (see [1,2]). It has also been suggested that nonlocal growth terms present a more realistic model in physics for compressible reactive gases. Problem (1.1) arises in the study of the flow of a fluid through a porous medium with an integral source (see [5,6]) and in the study of population dynamics (see [3,7]).

There have been many articles which deal with properties of solutions to local semilinear parabolic equations with homogeneous Dirichlet boundary condition (see [1,4,8] and references therein) and to a system of heat equations with nonlinear boundary condition (see [17] and references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [9,10]). In this case, the solution $u(x,t)$ describes entropy per volume of the material. The problem of nonlocal boundary conditions for linear parabolic equations of the type

$$\begin{cases} u_t - Au = 0, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\,dy, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.2}$$

with uniformly elliptic operator

$$A = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

and $c(x) \leqslant 0$ was studied by Friedman [11]. It was proved that the unique solution of (1.2) tends to 0 monotonically and exponentially as $t \to +\infty$ provided

$$\int_\Omega \bigl|\varphi(x,y)\bigr|\,dy \leqslant \rho < 1, \quad x \in \partial\Omega.$$

As for more general discussions on the dynamics of parabolic problem with nonlocal boundary conditions, one can see, e.g. [12] by Pao, where the following problem:

$$\begin{cases} u_t - Au = g(x,u), \quad (x,t) \in \Omega \times (0,T), \\ Bu(x,t) = \displaystyle\int_\Omega K(x,y)u(y,t)\,dy, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

was considered with

$$Bu = \alpha_0\frac{\partial u}{\partial \nu} + u$$

and recently Pao [13] gave the numerical solutions for diffusion equations with nonlocal boundary conditions.

Parabolic equations with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, the problem of the form

$$\begin{cases} u_t - \Delta u = \displaystyle\int_\Omega g(u)\,dx, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\,dy, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.4}$$

was studied by Lin and Liu [14]. They established local existence, global existence and nonexistence of solutions and discussed the blow-up properties of solutions.

In [16], Wang et al. studied the following problem

$$\begin{cases} u_t - \Delta u^m = g(u), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \int\limits_\Omega \varphi(x,y) u(y,t)\,dy, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

in the case of $g(u) = u^p$ with $m > 1$, $p > 1$. They obtained the blow-up condition and its blow-up rate estimate.

The above studies show that the growth or decay properties of the solution to problem (1.3)–(1.5) depend on the growth of the $g(u)$, which is similar to general semilinear equation with zero boundary condition. On the other hand, due to the appearance of the nonlocal boundary condition, the properties of solution heavily depend on the weight function $\varphi(x,y)$ as well.

The porous medium equation and the equations of porous medium type with local or nonlocal source and with local boundary conditions have been studied by a large number of authors since the 1970s in the last century (see [16,18,21,23] and references therein). Motivated by the above works, we are interested in the blow-up properties of problem (1.1). The aim of this paper is twofold. Firstly, we establish the global existence and finite time blow-up of the solution. Secondly, we establish the precise blow-up rate estimates for all solutions which blow up. Our main results could be stated as follows.

Theorem 1.1. *Suppose that $\int_\Omega \varphi(x,y)\,dy \geqslant 1$ for $x \in \partial\Omega$. If $p + q > 1$, then the solution of* (1.1) *blows up in finite time.*

Theorem 1.2. *Suppose that $\int_\Omega \varphi(x,y)\,dy < 1$ for $x \in \partial\Omega$.*

(1) *If $p < m - q$, then every nonnegative solution of* (1.1) *exists globally.*
(2) *If $p > m - q$, then the solution of* (1.1) *exists globally for sufficiently small $u_0(x)$, while it blows up in finite time for large initial data.*
(3) *If $p = m - q$ and a is sufficiently small, then the solution of* (1.1) *exists globally.*

To describe conditions for blow-up of solutions, we need the following assumptions on the initial data $u_0(x)$:

(H_1) $\Delta u_0^m(x) + a u_0^q \int_\Omega u_0^p(y,t)\,dy > 0$, for $x \in \Omega$;
(H_2) There exists a constant $\delta \geqslant \delta_0 > 0$, such that

$$\Delta u_0^m(x) + a u_0^q \int\limits_\Omega u_0^p(y,t)\,dy - \delta u_0^{p+q} \geqslant 0,$$

where δ_0 will be given later.

Theorem 1.3. *Suppose that $\int_\Omega \varphi(x,y)\,dy \leqslant 1$ and assumptions* (H_1)–(H_2) *hold. If the solution $u(x,t)$ blows up in finite time T, then there exist constants $C > c > 0$ such that*

$$c(T-t)^{-\frac{1}{p+q-1}} \leqslant \max_{x\in\overline{\Omega}} u(x,t) \leqslant C(T-t)^{-\frac{1}{p+q-1}}.$$

This paper is organized as follows. Section 2 deals with the maximum principle and comparison principle used for the model. Theorems 1.1 and 1.2 will be proved in Section 3. In Sections 4 and 5, the blow-up rate and profile will be discussed.

2. Comparison principle and local existence

In this section, we start with the definition of supersolution and subsolution of (1.1) and comparison theorem. Let $\Omega_T = \Omega \times (0,T)$ and $\Omega_T \cup \Gamma_T = \overline{\Omega} \times [0,T)$.

Definition 2.1. A function $\underline{u}(x,t)$ is called a subsolution of (1.1) on Ω_T, if $\underline{u}(x,t) \in C(\Omega_T \cup \Gamma_T) \cap C^{2,1}(\Omega_T)$ and satisfies

$$\begin{cases} \underline{u}_t \leqslant \Delta \underline{u}^m + a\underline{u}^q \int_\Omega \underline{u}^p(y,t)\,dy, & (x,t) \in \Omega \times (0,T), \\ \underline{u}(x,t) \leqslant \int_\Omega \varphi(x,y)\underline{u}(y,t)\,dy, & (x,t) \in \partial\Omega \times (0,T), \\ \underline{u}(x,0) \leqslant \underline{u}_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

A supersolution $\bar{u}(x,t)$ of (1.1) is defined analogously by the above inequalities with each inequality reversed.

A weak solution of (1.1) is a function which is both a subsolution and a supersolution of (1.1). The following comparison lemma plays a crucial role in our proof which can be obtained by similar arguments as in [14,20].

Theorem 2.2. *Suppose that $w(x,t) \in C(\Omega_T \cup \Gamma_T) \cap C^{2,1}(\Omega_T)$ satisfies*

$$\begin{cases} w_t - d(x,t)\Delta w \geqslant c_1(x,t)w + c_3(x,t)\int_\Omega c_2(x,t)w(x,t)\,dx, & (x,t) \in \Omega \times (0,T), \\ w(x,t) \geqslant \int_\Omega c_4(x,y)w(y,t)\,dy, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{2.2}$$

where $c_i(x,y)$ $(i=1,\ldots,4)$ are bounded functions and $c_2(x,y), c_3(x,y) \geqslant 0$ in Ω_T, $c_4(x,y) \geqslant 0$ for $x \in \partial\Omega$, $y \in \Omega$ and is not identically zero. Then $w(x,0) > 0$ for $x \in \overline{\Omega}$ implies $w(x,t) > 0$ in Ω_T. Moreover, if $c_4(x,y) \equiv 0$ or if $c_4(x,y) \geqslant 0$ and $\int_\Omega c_4(x,y)\,dy \leqslant 1$, $x \in \partial\Omega$, then $w(x,0) \geqslant 0$ for $x \in \overline{\Omega}$ implies $w(x,t) \geqslant 0$ in Ω_T.

Remark 2.3. The nonnegative of $c_4(x,y)$ plays an important role, see Remark 2.1 of [14]. If the nonlocal reaction source $\int_\Omega c_2(x,t)w(x,t)\,dx$ is replace by local reaction source $c_2(x,t)w(x,t)$, the nonnegativity of $c_3(x,t)$ is not necessary, see Theorem 2.1 of [20].

Theorem 2.4. *Let u and v be a nonnegative subsolution and supersolution, respectively, with $u(x,0) \leqslant v(x,0)$ for $x \in \overline{\Omega}$. Then, $u \leqslant v$ in Ω_T.*

Proof. Let $\psi(x,t) \in C^{2,1}(\Omega_T)$ be a nonnegative function with $\psi|_{\partial\Omega\times(0,T)} = 0$. Multiply the first inequality in (2.1) by $\psi(x,t)$ and then integrate it on Ω_T for any $0 < t < T$, we get

$$\int_\Omega u(x,t)\psi(x,t)\,dx \leqslant \int_\Omega u(x,0)\psi(x,0)\,dx + \iint_{Q_t} \left(u\psi_\tau + u^m \Delta\psi + a\psi u^q \int_\Omega u^p\,dy \right) dx\,d\tau$$
$$- \int_0^t \int_{\partial\Omega} \frac{\partial\psi}{\partial n} \left(\int_\Omega \varphi(x,y)u(y,\tau)\,dy \right)^m dS\,d\tau,$$

where n is the unit outward normal to the lateral boundary of Ω_T. On the other hand, the supersolution v satisfies the reversed inequality,

$$\int_\Omega v(x,t)\psi(x,t)\,dx \geqslant \int_\Omega v(x,0)\psi(x,0)\,dx + \iint_{Q_t} \left(v\psi_\tau + v^m \Delta\psi + a\psi v^q \int_\Omega v^p\,dy \right) dx\,d\tau$$

$$-\int_0^t\int_{\partial\Omega}\frac{\partial\psi}{\partial n}\bigg(\int_\Omega\varphi(x,y)v(y,\tau)\,dy\bigg)^m\,dS\,d\tau.$$

Set $w(x,t)=u(x,t)-v(x,t)$, we have

$$\int_\Omega w(x,t)\psi(x,t)\,dx\leqslant\int_\Omega w(x,0)\psi(x,0)\,dx+\iint_{Q_t}\big(\psi_\tau+\Phi_1\Delta\psi+a\psi\Phi_2(x,s)\big)w\,dx\,d\tau$$
$$-\int_0^t\int_{\partial\Omega}\frac{\partial\psi}{\partial n}m\xi^{m-1}\bigg(\int_\Omega\varphi(x,y)w(y,\tau)\,dy\bigg)\,dS\,d\tau,$$

where $\Phi_1(x,s)=\int_0^1 m(\theta u+(1-\theta)v)^{m-1}\,d\theta$, $\Phi_2(x,s)=q\int_\Omega u^p\,dx\int_0^1(\theta u+(1-\theta)v)^{q-1}\,d\theta+pv^q\int_\Omega(\int_0^1(\theta u+(1-\theta)v)^{p-1}\,d\theta)\,dx$, and ξ is a function between $\int_\Omega\varphi(x,y)u(y,\tau)\,dy$ and $\int_\Omega\varphi(x,y)v(y,\tau)\,dy$. Noticing that u,v are nonnegative bounded functions and $\frac{\partial\psi}{\partial n}\leqslant 0$ on $\partial\Omega$, we choose appropriate function ψ as in [22, pp. 118–123] to obtain

$$\int_\Omega w(x,t)_+\,dx\leqslant C_1\int_\Omega w(x,0)_+\,dx+C_2\int_0^t\int_\Omega w(y,\tau)_+\,dy\,d\tau\leqslant C_2\int_0^t\int_\Omega w(y,\tau)_+\,dy\,d\tau,$$

here we use $w(x,0)=u(x,0)-v(x,0)\leqslant 0$. By Gronwall's inequality, we have $w(x,t)\leqslant 0$. □

Remark 2.5. In [19] or [20], if u and v is subsolution and supersolution for the corresponding problem, then $u(x,0)<v(x,0)$ implies $u<v$, $x\in\Omega_T$. And when $\int_\Omega\varphi(x,y)\,dy\leqslant 1$, $u(x,0)\leqslant v(x,0)$ implies $u\leqslant v$, $x\in\Omega_T$. From Theorem 2.3, we know that $u(x,0)\leqslant v(x,0)$ implies $u\leqslant v$, $x\in\Omega_T$ and we have no restriction on $\varphi(x,y)$ here.

Theorem 2.6. *Let $\varphi(x,y)$ be a continuous function, u_0 be continuous on Ω and satisfies the compatibility condition $u_0(x)=\int_\Omega\varphi(x,y)u_0(y)\,dy>0$ for $x\in\partial\Omega$. Then there exists T $(0<T\leqslant\infty)$ and $u(x,t)\in C(\Omega_T\cup\Gamma_T)\cap C^{2,1}(\Omega_T)$, such that $u(x,t)$ is the unique maximal solution of* (1.1). *If $T<\infty$, we have* $\lim_{t\to T}\sup_{x\in\overline{\Omega}}u(\cdot,t)=+\infty$.

Local in time existence of positive classical solutions of the problem (1.1) can be obtained by using fixed point theorem (see [19]), the representation formula and the contraction mapping principle as in [14]. By the above comparison principle, we get the uniqueness of solution to the problem. The proof is more or less standard, so is omitted here.

3. Global existence and blow-up in finite time

Compared with usual homogeneous Dirichlet boundary data, the weight function $\varphi(x,y)$ plays a important role in the global existence or global nonexistence results for problem (1.1).

Proof of Theorem 1.1. Consider the equation

$$v'(t)=a|\Omega|v^{p+q},\qquad v(0)=v_0,\tag{3.1}$$

where $0<v_0<\min_{\overline{\Omega}}u_0(x)$. It is easy to see the solution of (3.1) is a subsolution of (1.1). Here we use the condition $\int_\Omega\varphi(x,y)\,dy\geqslant 1$. It is well known that the solution to (3.1) blows up in finite time if $p+q>1$. By the comparison theorem, we get the global nonexistence result of (1.1). □

From now on, we begin to consider the problem in the case of $\int_\Omega\varphi(x,y)\,dy<1$. In this case, the result depends on the comparison of the parameter p,q and m.

Let $\psi(x)$ be a unique positive solution of the following linear elliptic problem:

$$\begin{cases} -\Delta\psi(x) = \varepsilon_0, & x \in \Omega, \\ \psi(x) = \int_\Omega \varphi(x,y)\,dy, & x \in \partial\Omega, \end{cases} \tag{3.2}$$

where ε_0 is a positive constant such that $0 \leqslant \psi(x) \leqslant 1$ (as $\int_\Omega \varphi(x,y)\,dy < 1$, there exists such ε_0). Let $\max_{x\in\overline{\Omega}} \psi(x) = K_1$, $\min_{x\in\overline{\Omega}} \psi(x) = K_2$.

We define the function $w(x,t)$ as follows:

$$w(x,t) = M\psi^{\frac{1}{m}}(x), \tag{3.3}$$

where M is a constant to be determined later. Then, we have

$$w_t - \Delta w^m - aw^q \int_\Omega w^p\,dx = M^m\varepsilon_0 - aM^{p+q}\psi^{\frac{q}{m}} \int_\Omega \psi^{\frac{p}{m}}\,dx \geqslant M^m\varepsilon_0 - aM^{p+q}K_1^{\frac{p+q}{m}}|\Omega|. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned} w|_{\partial\Omega} &= M\biggl(\int_\Omega \varphi(x,y)\,dy\biggr)^{\frac{1}{m}} > M\int_\Omega \varphi(x,y)\,dy \geqslant M\int_\Omega \varphi(x,y)\psi^{\frac{1}{m}}\,dy \\ &= \int_\Omega \varphi(x,y)w(y,t)\,dy. \end{aligned} \tag{3.5}$$

Here we used $\int_\Omega \varphi(x,y)\,dy < 1$ and $0 \leqslant \psi(x) \leqslant 1$.

Hence, by (3.4) and (3.5), we get the following results.

Proof of Theorem 1.2. (1) In the case of $p < m - q$, by (3.4) and (3.5), we know if we choose M as

$$M = \max\Bigl\{\bigl(a|\Omega|K_1^{\frac{p+q}{m}}\varepsilon_0^{-1}\bigr)^{\frac{1}{m-p-q}}, K_2^{-\frac{1}{m}}\max_{x\in\overline{\Omega}} u_0(x)\Bigr\},$$

then $w(x,t)$ defined as (3.3) is a supersolution of (1.1). By Theorem 2.4, we know that $u(x,t) \leqslant w(x,t)$, then $u(x,t)$ exists globally.

(2) In the case of $p > m - q$, we have two different results. Firstly, we will prove the globally result, the proof is similarly as the above one. As far as inequality (3.4) is concerned, $M^m\varepsilon_0 - aM^{p+q}K_1^{\frac{p+q}{m}}|\Omega| = 0$ results in

$$M = \bigl(K_1^{-\frac{p+q}{m}}\bigl(a|\Omega|\bigr)^{-1}\varepsilon_0\bigr)^{\frac{1}{p+q-m}}. \tag{3.6}$$

Hence, w is a supersolution of (1.1) provided that

$$u_0(x) \leqslant w(x,t) \leqslant \bigl(K_1^{-\frac{p+q}{m}}\bigl(a|\Omega|\bigr)^{-1}\varepsilon_0\bigr)^{\frac{1}{p+q-m}}\bigl(\psi(x)\bigr)^{\frac{1}{m}}.$$

On the other hand, to prove the blow-up result, we consider the following well-known porous medium equation (see [21]):

$$\begin{cases} v_t = \Delta v^m + av^q \int_\Omega v^p\,dy, & (x,t) \in \Omega \times (0,T), \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3.7}$$

Let $v(x,t)$ be the solution of this equation. It is obviously that $v(x,t)$ is a subsolution of problem (1.1). It is known to all that $v(x,t)$ blows up in finite time if $u_0(x)$ is large enough. By Theorem 2.4, the solution of (1.1) blows up in finite time.

(3) Lastly, in case of $p = m - q$, choose $a < (K_1|\Omega|)^{-1}$, we have

$$w_t - \Delta w^m - a w^q \int_\Omega w^p \, dx \geqslant 0. \tag{3.8}$$

By (3.6) and (3.8), $w(x,t)$ is a supersolution of (1.1), then by Theorem 2.4, $w(x,t)$ exists globally. □

4. Blow-up rate estimates

Motivated by [21], in this section, we will show the blow-up rate of solution to problem (4.1), which gives the blow-up rate of $u(x,t)$ near the blow-up time immediately.

To obtain the estimate, we firstly introduce a transformation. Let $u^m = v$, then (1.1) becomes

$$\begin{cases} v_t = m v^r \left(\Delta v + a v^{q_1} \displaystyle\int_\Omega v^{p_1} \, dy \right), & (x,t) \in \Omega \times (0,T), \\ v(x,t) = \left(\displaystyle\int_\Omega \varphi(x,y) v^n(y,t) \, dy \right)^m, & x \in \partial\Omega, \ t > 0, \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $0 < r = (m-1)/m < 1$, $n = 1/m$, $p_1 = p/m$, $q_1 = q/m$, $v_0(x) = u_0^m(x)$.

Under this transformation, assumptions (H_1)–(H_2) become

(H_1') $\Delta v_0(x) + a v_0^{q_1} \int_\Omega v_0^{p_1}(y,t) \, dy > 0$, for $x \in \Omega$;
(H_2') There exists a constant $\delta' \geqslant \delta_1 > 0$, such that

$$\Delta v_0(x) + a v_0^{q_1} \int_\Omega v_0^{p_1}(y,t) \, dy - \delta' v_0^{p_1+q_1} \geqslant 0,$$

where δ_1 will be given later.

Suppose that the solution of (4.1) blows up in finite time T and set $V(t) = \max_{x \in \overline{\Omega}} v(x,t)$, then we have the following lemma.

Lemma 4.1. *Suppose that $v_0(x)$ satisfies (H_1')–(H_2'), then there exists a positive constant M_1, such that*

$$V(t) \geqslant M_1 (T-t)^{-1/(p_1+q_1+r-1)}. \tag{4.2}$$

Proof. We can easily see that $V(t)$ is Lipschitz continuous and thus it is differential almost everywhere.

By the first equation in (4.1) and $\Delta V(t) \leqslant 0$, we have [15, Theorem 4.5]

$$V'(t) \leqslant m a V^{r+q_1} \int_\Omega v^{p_1} \, dy \leqslant m a |\Omega| V^{r+p_1+q_1}(t).$$

Hence,

$$-\left(V^{1-r-p_1-q_1}(t)\right)' \leqslant m a |\Omega| (r + p_1 + q_1). \tag{4.3}$$

Integrating (4.3) over (t,T), we get

$$V(t) \geqslant M_1 (T-t)^{-1/(r+p_1+q_1-1)},$$

where $M_1 = [am(r + p_1 + q_1)|\Omega|]^{-1/(r+p_1+q_1-1)}$, we draw the conclusion. □

Lemma 4.2. *Suppose that $v_0(x)$ satisfies (H_1')–(H_2'), then there exists a constant $\delta'>0$, which is defined in (H_2'), such that*

$$v_t-\delta' v^{r+p_1+q_1}\geqslant 0,\quad (x,t)\in\Omega_T. \tag{4.4}$$

Proof. Set $J(x,t)=v_t-\delta' v^{r+p_1+q_1}$, a straightforward computation yields

$$\begin{aligned}
&J_t-mv^r\Delta J-\Bigl(2r\delta' v^{r+p_1+q_1-1}+maq_1v^{r+q_1-1}\int_\Omega v^{p_1}\,dx\Bigr)J-map_1v^{r+q_1-1}\int_\Omega v^{p_1-1}J\,dx\\
&\quad=rv^{-1}J^2+m\delta'(r+p_1+q_1)(r+p_1+q_1-1)v^{r+p_1+q_1-2}|\nabla v|^2+r\delta'^2v^{2r+2p_1+2q_1-1}\\
&\qquad+map_1\delta' v^{r+q_1}\int_\Omega v^{2p_1+q_1+r-1}\,dx-ma(p_1+r)\delta' v^{2p_1+q_1+2r-1}\int_\Omega v^{p_1}\,dx\\
&\quad\geqslant r\delta'^2v^{2r+2p_1+2q_1-1}+map_1\delta' v^{r+q_1}\int_\Omega v^{2p_1+q_1+r-1}\,dx-ma(p_1+r)\delta' v^{p_1+2q_1+2r-1}\int_\Omega v^{p_1}\,dx.
\end{aligned}\tag{4.5}$$

Since $p_1/(r+2p_1+q_1-1)+(r+p_1+q_1-1)/(r+2p_1+q_1-1)=1$ (notice $p+q>m$ implies $p_1+q_1>1$), by virtue of Young's inequality, we have

$$\begin{aligned}
&v^{p_1+q_1+r-1}\Bigl(\int_\Omega v^{2p_1+q_1+r-1}\,dx\Bigr)^{p_1/(2p_1+q_1+r-1)}\\
&\quad\leqslant\frac{p_1+q_1+r-1}{2p_1+q_1+r-1}\bigl(\theta v^{p_1+q_1+r-1}\bigr)^{(2p_1+q_1+r-1)/(p_1+q_1+r-1)}\\
&\qquad+\frac{p_1}{2p_1+q_1+r-1}\theta^{-(2p_1+q_1+r-1)/p_1}\int_\Omega v^{2p_1+q_1+r-1}\,dx,
\end{aligned}\tag{4.6}$$

where $\theta=((p_1+r)/(2p_1+q_1+r-1))^{p_1/(2p_1+q_1+r-1)}|\Omega|^{p_1(p_1+q_1+r-1)/(2p_1+q_1+r-1)^2}$.

Hölder's inequality implies

$$\int_\Omega v^{p_1}\,dx\leqslant|\Omega|^{(p_1+q_1+r-1)/(2p_1+q_1+r-1)}\Bigl(\int_\Omega v^{2p_1+q_1+r-1}\,dx\Bigr)^{p_1/(2p_1+q_1+r-1)}. \tag{4.7}$$

Using (4.6) and (4.7), then (4.5) becomes

$$\begin{aligned}
&J_t-mv^r\Delta J-\Bigl(2r\delta' v^{r+p_1+q_1-1}+maq_1v^{r+q_1-1}\int_\Omega v^{p_1}\,dx\Bigr)J-map_1v^{r+q_1-1}\int_\Omega v^{p_1-1}J\,dx\\
&\quad\geqslant\delta' r\delta' v^{2r+2p_1+2q_1-1}-ma(p_1+q_1+r-1)\theta^{(2p_1+q_1+r-1)^2/p_1(p_1+q_1+r-1)}v^{2p_1+2q_1+2r-1}\\
&\quad=r\delta'(\delta'-\delta_1)v^{2p_1+2q_1+2r-1}\geqslant 0.
\end{aligned}\tag{4.8}$$

Fix $(x,t)\in\partial\Omega\times(0,T)$, we have

$$\begin{aligned}
J(x,t)&=v_t-\delta' v^{r+p_1+q_1}\\
&=\Bigl(\int_\Omega\varphi(x,y)u(y,t)\,dy\Bigr)^{m-1}\Bigl(\int_\Omega m\varphi(x,y)u_t(y,t)\,dy-\delta'\Bigl(\int_\Omega\varphi(x,y)u(y,t)\,dy\Bigr)^{p+q}\Bigr).
\end{aligned}$$

Since $v_t(y,t)=J(y,t)+\delta' v^{r+p_1+q_1}$, we have

$$\begin{aligned}
&\int_\Omega m\varphi(x,y)u_t(y,t)\,dy-\delta'\Bigl(\int_\Omega\varphi(x,y)u(y,t)\,dy\Bigr)^{p+q}\\
&\quad=\int_\Omega m\varphi(x,y)v^{\frac{1-m}{m}}(y,t)J(y,t)\,dy+\delta'\Bigl(\int_\Omega\varphi(x,y)v^{\frac{p+q}{m}}(y,t)\,dy-\Bigl(\int_\Omega\varphi(x,y)v^{\frac{1}{m}}(y,t)\,dy\Bigr)^{p+q}\Bigr).
\end{aligned}\tag{4.9}$$

Noticing that $0 < F(x) = \int_\Omega \varphi(x, y)\,dy \leqslant 1$, $x \in \partial\Omega$, we can apply Jensen's inequality to the last integral in the above inequality,

$$\int_\Omega \varphi(x, y) v^{\frac{p+q}{m}}(y, t)\,dy - \left(\int_\Omega \varphi(x, y) v^{\frac{1}{m}}(y, t)\,dy\right)^{p+q}$$
$$\geqslant F(x)\left(\int_\Omega \varphi(x, y) v^{\frac{1}{m}}(y, t)\frac{dy}{F(x)}\right)^{p+q} - \left(\int_\Omega \varphi(x, y) v^{\frac{1}{m}}(y, t)\,dy\right)^{p+q} \geqslant 0.$$

Here, we used $p + q > 1$ and $0 < F(x) \leqslant 1$ in the last inequality. Hence $(x, t) \in \partial\Omega \times (0, T)$,

$$J(x, t) \geqslant \left(\int_\Omega \varphi(x, y) v^{\frac{1}{m}}(y, t)\,dy\right)^{m-1} + \int_\Omega m\varphi(x, y) v^{\frac{1-m}{m}}(y, t) J(y, t)\,dy. \tag{4.10}$$

On the other hand, (H_1')–(H_2') imply that $J(x, 0) \geqslant 0$. Combined inequalities (4.8)–(4.10), as v is a positive bounded continuous function for $(x, t) \in \overline{\Omega} \times [0, T)$, by Lemma 2.2, we know that $J(x, t) \geqslant 0$ for $(x, t) \in \overline{\Omega} \times [0, T)$, i.e. $v_t \geqslant \delta' v^{r+p_1+q_1}$. □

Integrating this inequality over (t, T), we have

$$v \leqslant \delta'(r + p_1 + q_1)^{-1/(r+p_1+q_1-1)}(T - t)^{-1/(r+p_1+q_1-1)}, \quad (x, t) \in \Omega \times (0, T). \tag{4.11}$$

Setting $M_2 = \delta'^{1/m}(r + p_1 + q_1)^{-1/(r+p_1+q_1)}$ and combining (4.2) and (4.11), we obtain the following result.

Lemma 4.3. *Suppose that $v_0(x)$ satisfies (H_1')–(H_2') and the solution of problem (4.1) blows up in finite time. Then there exist two positive constants M_1, M_2 such that*

$$M_1(T - t)^{-1/(r+p_1+q_1-1)} \leqslant V(t) \leqslant M_2(T - t)^{-1/(r+p_1+q_1-1)},$$

where T is the blow-up time of $v(x, t)$.

Noticing that $v = u^m$, we get the following inequalities immediately

$$c(T - t)^{-1/(p+q-1)} \leqslant u(x, t) \leqslant C(T - t)^{-1/(p+q-1)}, \tag{4.12}$$

where $c = M_1^{1/m}$, $C = M_2^{1/m}$, so we obtain the blow-up rate estimate in Theorem 1.3.

Remark. From Theorem 1.3, we know that in the case of $\int_\Omega \varphi(x, y)\,dy \leqslant 1$, $x \in \partial\Omega$, the blow-up rate of porous medium equation with nonlocal boundary condition is the same as that of general porous medium equation (3.7).

5. Blow-up profile

Throughout this section, we assume that $m = 1$, $q < 1$ and the solution $u(x, t)$ of (1.1) blows up in finite time T. We use the notation $v \sim w$ for $\lim_{t\to T} v(t)/w(t) = 1$.

Set

$$g(t) = a\int_\Omega u^p\,dx, \qquad G(t) = \int_0^t g(s)\,ds.$$

As in [21,24], to show the local solvability of (1.1), we consider the similarity regularized problem. Using the Schauder's fixed point theorem, we can prove that the regularized problem admits a unique classical solution $u_\varepsilon(x, t)$ on Ω_{T_ε}.

Lemma 5.1. *Assume that $\Delta u_0 \leqslant 0$ on $\overline{\Omega}$, $\varphi(x, y) \geqslant 0$ for $(x, y) \in \partial\Omega \times \Omega$, $\int_\Omega \varphi(x, y)\,dy \leqslant c < 1$ for $x \in \partial\Omega$. Then $\Delta u \leqslant 0$ on any compact subset of Ω.*

Proof. Let u_ε be the solution of (4.1) on Ω_{T_ε}. Set $W = \Delta u_\varepsilon$, it follows from (4.1) in Ω_{T_ε} that

$$W_t = \Delta\left(\Delta u_\varepsilon + a u_\varepsilon^q \int_\Omega u_\varepsilon^p\,dx\right) = \Delta W + aq(q-1)u_\varepsilon^{q-2}|\nabla u_\varepsilon|^2 \int_\Omega u_\varepsilon^p\,dx + aqu_\varepsilon^{q-1}\Delta u_\varepsilon \int_\Omega u_\varepsilon^p\,dx.$$

In view of $u_\varepsilon > 0$, $q < 1$, we have

$$W_t - \Delta W \leqslant aqu_\varepsilon^{q-1} W \int_\Omega u_\varepsilon^p\,dx. \tag{5.1}$$

On the other hand, let $C_1 = \max_{x\in\Omega_{T_\varepsilon}} u_\varepsilon$, $C_2 = \min_{x\in\Omega_{T_\varepsilon}} u_\varepsilon$, for $(x,t) \in \partial\Omega_{T_\varepsilon} \times (0,T)$, then

$$\begin{aligned}
W(x,t) &= u_{\varepsilon t}(x,t) - u_\varepsilon^q \int_\Omega u_\varepsilon^p = \int_\Omega \varphi(x,y)u_{\varepsilon t}(y,t)\,dy - u_\varepsilon^q \int_\Omega u_\varepsilon^p \\
&= \int_\Omega \varphi(x,y)W(y,t)\,dy + \left(\int_\Omega \varphi(x,y)u_\varepsilon^q\,dy - u_\varepsilon^q\right)\int_\Omega u_\varepsilon^p \\
&\leqslant \int_\Omega \varphi(x,y)W(y,t)\,dy + \left(C_1^q \int_\Omega \varphi(x,y)\,dy - C_2^q\right)\int_\Omega u_\varepsilon^p \\
&\leqslant \int_\Omega \varphi(x,y)W(y,t)\,dy.
\end{aligned}$$

Otherwise, $W(x,0) = \Delta u_0 \leqslant 0$, $x \in \overline{\Omega}$. Then $W = \Delta u_\varepsilon \leqslant 0$. Therefore, $\Delta u \leqslant 0$ on any compact subset of Ω. □

Lemma 5.2. *Under the same conditions of Lemma* 5.1*, it holds that*

$$\lim_{t\to T} g(t) = \lim_{t\to T} G(t) = +\infty.$$

Proof. From Lemma 5.1, we have

$$u_t \leqslant u^q g(t). \tag{5.2}$$

Integrating (5.2) from 0 to t, we get

$$\frac{1}{1-q}u^{1-q}(x,t) \leqslant \int_0^t g(s)\,ds + \frac{1}{1-q}u^{1-q}(x,0). \tag{5.3}$$

Since $\lim_{t\to T}\|u\| = +\infty$ and $1-q>0$, we know that

$$\lim_{t\to T}\int_0^t g(s)\,ds = \lim_{t\to T} G(t) = +\infty. \qquad \square$$

Lemma 5.3. *Under the conditions of Lemma* 5.1*, we have*

$$\lim_{t\to T}\frac{\int_0^t G^{(1-q)/(1-p-q)}(s)\,ds}{G(t)} = 0. \tag{5.4}$$

Proof. We know from Theorem 1.3 that

$$\int_0^t G^{(1-q)/(1-p-q)}\,ds \leqslant C\int_0^t (T-s)^{1-q}\,ds.$$

On the other hand, by (5.3) and Theorem 1.3, we have

$$G(t) \geqslant cu^{1-p-q} \geqslant c(T-t).$$

In summary, we get the desired result. □

Lemma 5.4. *Under the conditions of Lemma* 5.1*, then*

$$\lim_{t\to T} \frac{u^{1-q}(x,t)}{(1-q)G(t)} = \lim_{t\to T} \frac{\|u(\cdot,t)\|_\infty^{1-q}}{(1-q)G(t)} = 1, \tag{5.5}$$

hold on any compact set of Ω.

Proof. It follows from the proof of Lemma 5.2 that

$$\limsup_{t\to T} \frac{u^{1-q}(x,t)}{(1-q)G(t)} \leqslant 1. \tag{5.6}$$

Next let us show the opposite inequality. Denote

$$z(x,t) = G(t) - \frac{u^{1-q}(x,t)}{1-q}, \qquad \beta(t) = \int_\Omega z(y,t)\varphi(y)\,dy, \tag{5.7}$$

where φ satisfies

$$-\Delta\varphi(x) = \lambda\varphi(x), \quad x \in \Omega, \qquad \varphi(x) = 0, \quad x \in \partial\Omega,$$

and $\int_\Omega \phi(x) = 1$. A direct computation shows that

$$\begin{aligned}\beta'(t) &= \int_\Omega \big(g(t) - u^{-q}(y,t)u_t(y,t)\big)\varphi(y)\,dy = -\int_\Omega u^{-q}(x,y)\Delta u(y,t)\varphi(y)\,dy\\ &\leqslant \lambda_1 \int_\Omega u^{1-q}(x,y)\varphi(y)\,dy \leqslant C\int_\Omega \big(G(t) - z(y,t)\big)\varphi(y)\,dy \leqslant C\bigg(G(t) + \int_\Omega z^-(y,t)\varphi(y)\,dy\bigg),\end{aligned}$$

where $z^- = \max\{-z, 0\}$. From (5.3), we know that for any $t \in (0,T)$,

$$\inf_\Omega z(x,t) \geqslant -C_1,$$

which means $z^- \leqslant C_1$ for $(x,t) \in \Omega \times (0,T)$, and thus,

$$\beta'(t) \leqslant CG(t) + C'. \tag{5.8}$$

Integrating (5.8) from 0 to t, we have

$$\beta(t) \leqslant C\bigg(1 + \int_0^t G(s)\,ds\bigg), \tag{5.9}$$

which implies

$$\int_\Omega \big|z(y,t)\big|\varphi(y)\,dy \leqslant C\bigg(1 + \int_0^t G(s)\,ds\bigg). \tag{5.10}$$

Define $K_\rho = \{y \in \Omega\colon \operatorname{dist}(y,\partial\Omega) > \rho\}$. Since $-\Delta z \leqslant 0$ in $\Omega \times (0,T)$, by Lemma 4.5 in [1], we get

$$\sup_{K_\rho} z(x,t) \leqslant \frac{C}{\rho^{N+1}} C\bigg(1 + \int_0^t G(s)\,ds\bigg). \tag{5.11}$$

It follows from (5.3) and (5.12) that

$$-\frac{C}{G(t)} \leqslant 1 - \frac{u^{1-q}(x,t)}{(1-q)G(t)} \leqslant \frac{C}{\rho^{N+1}} \frac{C(1+\int_0^t G(s)\,ds)}{G(t)}, \tag{5.12}$$

for $x \in K_\rho$ and $t \in (0,T)$. Combining Lemmas 5.2 and 5.3 with (5.12), we get the desired result. □

Lemma 5.4 implies that the blow-up set is the whole domain Ω under the assumptions of Lemma 5.1.

From Lemma 5.4, we know,

$$G'(t) = a \int_\Omega u^p\,dx \sim a|\Omega|\big[(1-q)G(t)\big]^{p/(1-q)}, \quad \text{as } t \to T,$$

and hence

$$G(t) \sim (1-q)^{-1} M_1^{1-q} (T-t)^{(q-1)/(p+q-1)}, \quad \text{as } t \to T,$$

in which $M_1 = [a|\Omega|\frac{p+q-1}{1-q}]^{-1/(p+q-1)}(1-q)^{-1/(p+q-1)}$. Therefore, we get the following theorem.

Theorem 5.5. *Assume that the conditions of Lemma* 5.1 *are satisfied, we have*

$$\lim_{t\to T} (T-t)^{1/(p+q-1)} u(x,t) = M_1.$$

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